Finite Elements in Ocean Modelling

Vincent Legat, Emmanuel Hanert, Sébastien Legrand, Daniel Leroux, Eric Deleersnijder

Centre for Systems Engineering and Applied Mechanics
Institut d’Astronomie et de Géophysique G. Lemaître
Université Catholique de Louvain, Belgium

Département de Mathématiques et de Statistiques
Université Laval, Quebec, Canada
Unstructured grids in Ocean Modelling

**Finite Differences**
- Easy to handle
- Large experience

First Generation Models
K. Bryan (1969)

**Finite Elements**
**Finite Volumes**
- Mesh adaptivity
- Geometrical flexibility

Second Generation Models
Important issues:

- Advection schemes
- Basis functions
- Interoperability
- Solvers
- Subgrid scale phenomena
- Vertical discretization
Strategy of our work:

Linear Shallow Water Equations

\begin{align*}
\frac{\partial \eta}{\partial t} + H \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) &= 0 \\
\frac{\partial u}{\partial t} -fv + g \frac{\partial \eta}{\partial x} &= 0 \\
\frac{\partial v}{\partial t} +fu + g \frac{\partial \eta}{\partial y} &= 0
\end{align*}

Very crude model for geophysical flows, but allows the existence of inertia-gravity waves

Let’s see what has been done in Finite Differences and try to do better with Finite Elements …

Advection-Diffusion Equation

\[ \frac{\partial s}{\partial t} + \nabla \cdot \sigma = f, \]

Ocean water density depends on temperature and salinity
Finite Element for Shallow Water Equations

Appearance of Spurious Modes for the discrete gradient (Q) or discrete Coriolis (F) and discrete divergence (D) operators.

\( P \) is an elevation (or pressure) spurious mode if

\[ Q.P = 0 \]  with non constant \( P \).

\( U \) is a velocity spurious mode if

\[ F.U = 0 \]  and \( D.U = 0 \)  with non vanishing \( U \).
How to select a mixed element?

Finite Element Equivalents of Arakawa’s A, B and C grids

Spurious velocity modes seem to be « less annoying » from a physical point of view : some diffusion exists, but must be controlled by physics and not by numerics!
Our best selection

Non-conforming Discontinuous Linear Velocity
Continuous Linear Elevation

- No spurious modes in velocity or elevation
- Waves well propagated
- Cheap!

Crouzeix-Raviart (1973)
Leroux (2003)
Strategy of our work:

Linear Shallow Water Equations

\[
\frac{\partial \eta}{\partial t} + H \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0
\]
\[
\frac{\partial u}{\partial t} - f v + g \frac{\partial \eta}{\partial x} = 0
\]
\[
\frac{\partial v}{\partial t} + f u + g \frac{\partial \eta}{\partial y} = 0
\]

Let’s see what has been done in Finite Differences and try to do better with Finite Elements …

Advection-Diffusion Equation

\[
\frac{\partial s}{\partial t} + \nabla \cdot \sigma = f,
\]

- Very crude model for geophysical flows, but allows the existence of inertia-gravity waves
- Ocean water density depends on temperature and salinity
Ocean circulation strongly depends on water density gradients.

- Ocean water density depends on temperature and salinity:
  \[ \rho = \rho(S, T) \]

- Temperature and salinity are scalar quantities transported by the flow.
- An accurate representation of transport processes is thus required.
Typical Advection Diffusion Boundary Value Problem

The so-called Stommel Flow can be interpreted as the circulation in a close basin with a western boundary layer.

Find \( s(x,t) \) such that

\[
\frac{\partial s}{\partial t} + \nabla \cdot (us) = \nabla \cdot (k \nabla s), \quad \text{in } \Omega, \forall t,
\]

\[
k\nabla s \cdot n = 0, \quad \text{on } \partial \Omega,
\]

\[
s(x,0) = s_0(x), \quad \text{in } \Omega,
\]

with \( u \) a given divergence-free velocity such that \( u \cdot n = 0 \) on \( \partial \Omega \).
Mixing continuous and discontinuous approximations

Divide the domain into open subdomains where the approximation is always continuous

\[ \Omega = \bigcup_{i=1}^{N_{\Omega}} \Omega_i \quad \text{and} \quad \Omega_i \cap \Omega_j = \emptyset \quad \text{for} \ i \neq j. \]

\[ \Gamma = \bigcup_{l=1}^{N_{\Gamma}} \Gamma_l \quad \text{and} \quad \Gamma_l \cap \Gamma_m = \emptyset \quad \text{for} \ l \neq m. \]

To handle in a unified way…
- Continuous approximations
- Discontinuous approximations
- Mixed continuous and discontinuous approximations

Build a triangulation of the domain such that each element belongs to an unique subdomain.

\[ \Omega = \bigcup_{e=1}^{N_E} \overline{E}_e \quad \text{and} \quad E_e \cap E_f = \emptyset \quad \text{for} \ e \neq f, \]
Why to use discontinuous approximations for continuous exact solutions?

Because, one can obtain a better accuracy!
Uniform distribution is a stupid idea!

But, the cost of discontinuous dof is quite cheaper in most applications!

Same number of degrees of freedom (20) and same accuracy!
Sometimes, discontinuous approximation is better...

Jumps are useful when the boundary layer typical size is greater than the mesh size.

Discontinuous approximation collapses to the continuous one.
Deriving a weak variational discontinuous formulation

Find \( s(x, t) \in S = \{ v \in L^2(\Omega): v|_{\Omega_i} \in H^1(\Omega_i), \forall \Omega_i \in \mathcal{P} \} \) such that

\[
\sum_{i=1}^{N_{\Omega}} \int (\frac{\partial s}{\partial t} - s \mathbf{u} \cdot \nabla \hat{s} + k \nabla s \cdot \nabla \hat{s}) \, d\Omega + \sum_{i=1}^{N_{\Omega}} \int (s \mathbf{n} + k \nabla s \cdot \mathbf{n}) \hat{s} \, d\Gamma \\
+ \sum_{i=1}^{N_{\partial \Omega}} \int ([s] a(\hat{s})) \, d\Gamma + \sum_{i=1}^{N_{\partial \Omega}} \int ([k \nabla s \cdot \mathbf{n}] b(\hat{s})) \, d\Gamma = 0
\]

\( \forall \hat{s} \in S. \)

**Penalty term to enforce weak continuity of the solution**

\( a(\hat{s}) = \frac{k}{k} \hat{s} + \frac{1}{2} \mathbf{u} \cdot \mathbf{n} (\text{sign}(\mathbf{u} \cdot \mathbf{n}) - 1) \hat{s} \)

**Penalty term to enforce weak continuity of the flux**

\( b(\hat{s}) = \frac{1}{2} \hat{s} \)
After some tedious algebra...

Considering only once integrals along internal segments.

Find $s(x,t) \in S$ such that

$$
\sum_{i=1}^{N_\Omega} \int_{\Omega_i} \left( \frac{\partial s}{\partial t} - s u \cdot \nabla s + k \nabla s \cdot \nabla s \right) \, d\Omega
+ \sum_{i=1}^{N_\Gamma} \int_{\Gamma_i} \left( \langle s u \cdot n \rangle_s [\hat{s}] - \langle k \nabla s \cdot n \rangle [\hat{s}] + \frac{k}{h} [s][\hat{s}] \right) \, d\Gamma = 0
$$

$\forall \hat{s} \in S$,

$$
[f] = f|_{\Omega_i} - f|_{\Omega_j},
\langle f \rangle = \frac{1}{2} (f|_{\Omega_i} + f|_{\Omega_j}),
\langle f \rangle_s = \frac{1}{2} \left( (1 + \text{sign}(u \cdot n)) f|_{\Omega_i} + (1 - \text{sign}(u \cdot n)) f|_{\Omega_j} \right).
$$
Selecting shape functions...

**Usual Linear Shape Functions**

**Non-conforming Discontinuous Linear Shape Functions**

It is a discontinuous approximation, but the value at the mid-edge is common.

Such an element appears to be a very good compromise between continuous and discontinuous choice for advection-diffusion problem.

**Basic requirements**
- Robustness
- Cheap

**Continuous Linear**

**Discontinuous Usual Linear**
It looks like a variational crime, but it is not a crime!

Find $s^h(x, t) \in S^h$ such that

$$
\sum_{i=1}^{N_\Omega} \int_{\Omega_i} \left( \frac{\partial s^h}{\partial t} \bar{s}^h - s^h \mathbf{u} \cdot \nabla \bar{s}^h + k \nabla s^h \cdot \nabla \bar{s}^h \right) \, d\Omega
+ \sum_{l=1}^{N_{\Gamma_i}} \int_{\Gamma_l} \left( \langle s^h \mathbf{u} \cdot \mathbf{n} \rangle_{[\bar{s}^h]} - \langle k \nabla \bar{s}^h \cdot \mathbf{n} \rangle_{[\bar{s}^h]} + \frac{k}{h} [s^h]_{[\bar{s}^h]} \right) \, d\Gamma = 0
$$

$\forall \bar{s}^h \in S^h,$

It can be demonstrated that this term is exactly equal to zero for non-conforming linear shape functions.

By construction, a weak continuity is automatically achieved for non-conforming linear shape functions.

$$
\int_{\Gamma_l} [s^h] = 0
$$
If you like calculus...
\[ \int_{\Omega} \left( \phi_i \frac{\partial s^h}{\partial t} - \phi_i f - \nabla \phi_i \cdot \sigma \right) \, d\Omega + \int_{\partial \Omega} \phi_i \sigma \cdot n \, d\Gamma = 0 \quad \forall i \in I_e \]

The integral on \( \Omega \) may be decomposed into integrals on \( \Omega_e \) and \( \Omega \setminus \Omega_e \)

\[ \int_{\Omega_e} \left( \phi_i \frac{\partial s^h}{\partial t} - \phi_i f - \nabla \phi_i \cdot \sigma \right) \, d\Omega + \int_{\Omega \setminus \Omega_e} \left( \phi_i \frac{\partial s^h}{\partial t} - \phi_i f - \nabla \phi_i \cdot \sigma \right) \, d\Omega = 0 \quad \forall i \in I_e \]

Both terms can be interpreted as the integral of \( \sigma \cdot n \phi_i \) along the common interface

\[ \int_{\Omega_e} \left( \phi_i \frac{\partial s^h}{\partial t} - \phi_i f - \nabla \phi_i \cdot \sigma \right) \, d\Omega + \int_{\partial (\Omega \setminus \Omega_e) \cap \Omega_e} \phi_i \sigma \cdot n \, d\Gamma = 0 \quad \forall i \in I_e \]

\[ \int_{\partial \Omega_e} \phi_i \sigma \cdot n \, d\Gamma \]

---

**About conservation**

\[ \frac{d}{dt} \int_{\Omega_e} s^h \, d\Omega = \int_{\Omega_e} f \, d\Omega - \int_{\partial \Omega_e} \sigma \cdot n \, d\Gamma. \]
About dispersion

\[ \frac{\omega h}{u} = -2i - ie^{-ikh} \pm \sqrt{2 - 10e^{-ikh} - e^{-2ikh}}. \]
Finite Volumes and Finite Elements

\[ S \simeq S^h = \sum_{i=1}^{NV} S_i \phi_i, \text{ with } \phi_i \text{ piecewise linear} \]
Numerical example

\[ \Psi(x, y) = \frac{FL}{\pi \gamma \rho H} \cos\left(\frac{\pi y}{L}\right)(pe^{z+x} + qe^{z-x} - 1) \]

\[ s_0(x, y) = 10 \exp\left(-\frac{(x - 2L/3)^2 + (y - L/3)^2}{2(L/12.5)^2}\right). \]
Pure Advection
$e_{L_2} = \frac{\| s^r - s^h \|_{L_2(\Omega)}}{\| s^r \|_{L_2(\Omega)}}$

$e_{\min} = \min_{\Omega} s^h - \min_{\Omega} s^r$

$e_{\max} = \max_{\Omega} s^h - \max_{\Omega} s^r$
<table>
<thead>
<tr>
<th>Method</th>
<th>Image</th>
</tr>
</thead>
<tbody>
<tr>
<td>Finite Volumes</td>
<td>![Finite Volumes Image]</td>
</tr>
<tr>
<td>Continuous Linear</td>
<td>![Continuous Linear Finite Elements Image]</td>
</tr>
<tr>
<td>Discontinuous Linear</td>
<td>![Discontinuous Linear Finite Elements Image]</td>
</tr>
<tr>
<td>Non-conforming</td>
<td>![Non-conforming Linear Finite Elements Image]</td>
</tr>
</tbody>
</table>

Advection

Diffusion
Results are now slightly better for the continuous finite element and finite volume schemes. Ripples have decreased but not disappeared.

Discontinuous and nonconforming finite elements still give good results and the largest downward ripples are only 5% as large as the initial peak amplitude. As diffusion increases, all schemes tend to give similar results.
Conclusions

Today…
• A new specialized mesh generator tool has been developed
• Interpolation tools between several models and grids has been developed
• Preliminary calculations with simplified models have been performed and appears very promising
• Selection of suitable Finite Element approximations has been analyzed

Now, work has to be performed to obtain a realistic efficient and competitive prediction tool.
The so-called rotating cone...
Human influence on climate?
Intergovernmental Panel on Climate Change

1995
• The balance of evidence suggests a discernable human influence on global climate

2001
• An increasing body of observations gives a collective picture of a warming world and other changes in the climate system
• There is new and stronger evidence that most of the warming observed over the last 50 years is attributable to human activities
• The climate is expected to continue to change in the future
Climate System

= Atmosphere + Ocean + Cryosphere + Biosphere …

Ocean modelling is important because:
1. Ocean transports to the poles as much heat as the atmosphere
2. Ocean smoothes seasonal weather variations
3. Ocean and atmosphere exchange gases at the air-water interface
4. Ocean is a very large CO₂ sink